19 Transformando Integral Tripla Cartesiana em Cilíndrica

Reescreva a integral abaixo em coordenadas cilíndricas e calcule seu valor:

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} \boxed{z^2\,dz\,dx\,dy}$$

Solução: Montagem da integral tripla cilíndrica

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} \boxed{f(r,\theta,z)\,dz\,r\,dr\,d\theta}$$

Passo 1: Reescrevendo a função a ser integrada.

$$\boxed{f(x,y,z)=z^2} \longrightarrow \boxed{f(r,\theta,z)=z^2}$$

Passo 2: Reescrever limites de z com $g(r,\theta)$

- Inferior: $\sqrt{x^2+y^2}$
- Superior: $\sqrt{8-x^2-y^2}$

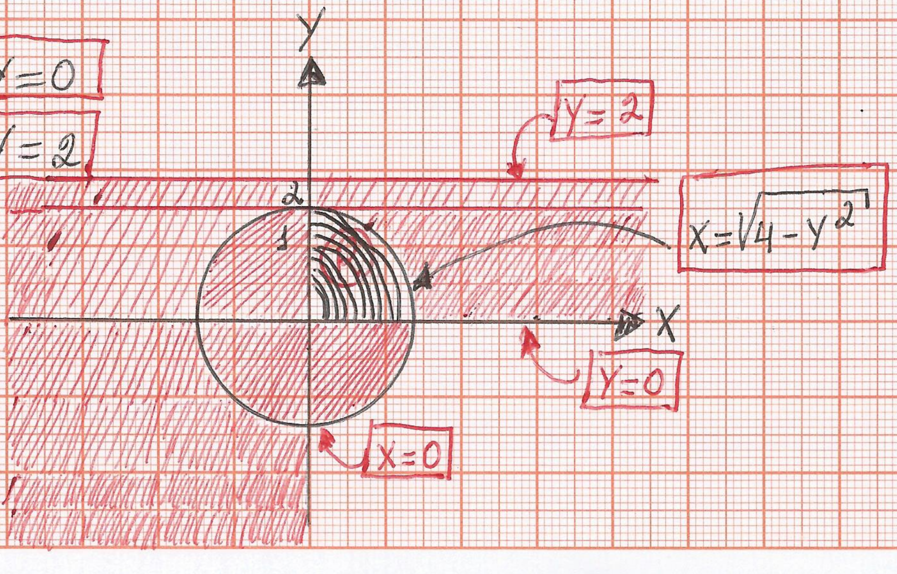
Usando $x^2+y^2=r^2$, temos: $z=\sqrt{r^2}$ → $\boxed{z=r}$

$\boxed{z=\sqrt{8-r^2}}$

Passo 3: Projeção do objeto no Plano XY.

- Limite Inferior de X: $\boxed{x=0}$
- Limite Superior de x: $\boxed{x=\sqrt{4-y^2}}$ → $x^2=4-y^2$ → $\boxed{x^2+y^2=4}$

- Limite Inferior de Y: $\boxed{y=0}$
- Limite Inferior de Y: $\boxed{y=2}$



Esboço:

A região Hachurada em vermelho não faz parte do nosso objeto!

Limites de r e θ :

$$\boxed{0 \le r \le 2}$$

$$\boxed{0 \le \theta \le \frac{\pi}{2}}$$

Agora vamos levar estes valores dos limites para as nossas integrais em coordenadas cilíndricas. Assim temos, a seguinte integral tripla cilíndrica:

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$\sqrt{8-0^2} \Rightarrow \sqrt{8}$ ← p/ $r=0$

$$\int_0^{\pi/2} \int_{r=0}^{r=2} \int_r^{\sqrt{8-r^2}} z^2 \, dz \, r \, dr \, d\theta$$

Adotamos o valor de zero para r, ou seja, $r=0$.

θ r z

Então temos: $\displaystyle\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{8}} z^2 \underbrace{dz\, dx\, dy}_{dV} \Rightarrow$

$$\Rightarrow \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{8}} (\rho \cdot \cos\phi)^2 \cdot \rho^2 \cdot \text{sen}\,\phi \, d\rho \, d\phi \, d\theta \Rightarrow$$

θ ϕ ρ

$$\Rightarrow \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{8}} \rho^2 \cdot \cos^2\phi \cdot \rho^2 \cdot \text{sen}\,\phi \, d\rho \cdot d\phi \, d\theta \Rightarrow$$

θ ϕ ρ

$$\Rightarrow \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{8}} \rho^4 \cdot \cos^2\phi \cdot \text{sen}\,\phi \, d\rho \cdot d\phi \, d\theta \Rightarrow$$

θ ϕ ρ

$$\Rightarrow \int_0^{\pi/2} \int_0^{\pi/4} \cos^2\phi \cdot \text{sen}\,\phi \left[\frac{\rho^{4+1}}{4+1} \right]_0^{\sqrt{8}} \Rightarrow dy\, dz$$

θ ϕ

$$\Rightarrow \int_0^{\pi/2} \int_0^{\pi/4} \cos^2\phi \cdot \text{sen}\,\phi \left[\frac{\rho^5}{5} \right]_0^{\sqrt{8}} \Rightarrow dy\, dz$$

θ ϕ

$$\int_0^{\pi/2}\int_0^{\pi/4} \cos^2\phi \cdot \operatorname{sen}\phi \left[\left(\frac{(\sqrt{8})^5}{5} - \cancel{\frac{0^5}{5}}\right)\right] \Rightarrow$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/4} \cos^2\phi \cdot \operatorname{sen}\phi \ \frac{(\sqrt{8})^5}{5} \Rightarrow$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/4} \cos^2\phi \cdot \operatorname{sen}\phi \ \frac{\sqrt{8}\cdot\sqrt{8}\cdot\sqrt{8}\cdot\sqrt{8}\cdot\sqrt{8}}{5} \Rightarrow dydz$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/4} \cos^2\phi \cdot \operatorname{sen}\phi \ \frac{\sqrt{2^3}\cdot\sqrt{2^3}\cdot\sqrt{2^3}\cdot\sqrt{2^3}\cdot\sqrt{2^3}}{5} \Rightarrow dydz$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/4} \cos^2\phi \cdot \operatorname{sen}\phi \ \frac{2\cdot\sqrt{2}\cdot 2\sqrt{2}\cdot 2\sqrt{2}\cdot 2\sqrt{2}\cdot 2\sqrt{2}}{5} \Rightarrow dydz$$

x x x x

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/4} \cos^2\phi \cdot \operatorname{sen}\phi \ \frac{2^5\cdot(\sqrt{2})^4\cdot\sqrt{2}}{5} \Rightarrow dy\,dz$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/4} \cos^2\phi \cdot \operatorname{sen}\phi \ \frac{32\cdot 2^{4/2}\cdot\sqrt{2}}{5} \Rightarrow dy\,dz$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/4} \cos^2\phi \cdot \operatorname{sen}\phi \ \frac{32\cdot 4\cdot\sqrt{2}}{5} \Rightarrow dy\,dz$$

(4)

$$\int_0^{\pi/2}\int_0^{\pi/4} \cos^2\phi \cdot \operatorname{sen}\phi \boxed{\frac{128\sqrt{2}}{5}}\, dy\, dz$$

θ ϕ

Aplicando a integração por substituição, temos: $u = \cos(\phi)$

$$\int_1^{\frac{\sqrt{2}}{2}} -u^2\,du \Rightarrow -\int_{\frac{\sqrt{2}}{2}}^{1} -u^2\,du \Rightarrow -\left[-\left(\frac{u^{2+1}}{2+1}\right)\right]_{\frac{\sqrt{2}}{2}}^{1} \Rightarrow$$

$$\Rightarrow \left[\frac{u^3}{3}\right]_{\frac{1}{\sqrt{2}}}^{1} \Rightarrow \left[\left(\frac{1^3}{3} - \frac{(1/\sqrt{2})^3}{3}\right)\right] \Rightarrow$$

$$\frac{\sqrt{2}\cdot\sqrt{2}}{2\cdot\sqrt{2}} = \frac{2}{2\sqrt{2}}$$

$$\boxed{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \left[\frac{1}{3} - \frac{1/2\sqrt{2}}{3}\right] \Rightarrow \left[\frac{1}{3} - \frac{1}{2\sqrt{2}}\cdot\frac{1}{3}\right] \Rightarrow$$

$$\Rightarrow \boxed{\left[\frac{1}{3} - \frac{1}{6\sqrt{2}}\right]}$$

Então temos:

$$\Rightarrow \int_0^{\pi/2} \left(\frac{1}{3} - \frac{1}{6\sqrt{2}}\right)\cdot\frac{128\sqrt{2}}{5}\ 1\, dz \Rightarrow$$

θ

$$\Rightarrow \int_0^{\pi/2} \left(\frac{1}{3} - \frac{1}{6\sqrt{2}}\right)\cdot\frac{128\sqrt{2}}{5}\left[z\right]_0^{\pi/2} \Rightarrow$$

θ

$$\Rightarrow \int_0^{\pi/2} \left(\frac{1}{3} - \frac{1}{6\sqrt{2}}\right)\cdot\frac{128\sqrt{2}}{5}\left[\frac{\pi}{2} - \phi\right] \Rightarrow \boxed{\frac{\pi}{2}}$$

θ

5

$$\rightarrow \left(\frac{1}{3} - \frac{1}{6\sqrt{2}}\right) \cdot \frac{\overset{:2}{128}\sqrt{2}}{5} \cdot \frac{\pi}{2\,:2} \rightarrow$$

$$\rightarrow \left(\frac{1}{3} - \frac{1}{6\sqrt{2}}\right) \cdot \frac{64\sqrt{2}}{5} \cdot \pi \rightarrow$$

$$\rightarrow \left(\frac{1}{3} - \frac{1}{6\sqrt{2}}\right) \cdot \left(\frac{\pi \cdot 64\sqrt{2}}{5}\right) \rightarrow$$

$$\rightarrow \frac{\pi\, 64\sqrt{2}}{15} - \frac{\pi\, 64\cancel{\sqrt{2}}}{30\cancel{\sqrt{2}}} \rightarrow$$

$$\rightarrow \frac{\pi\, 64\sqrt{2}}{15} - \frac{\pi\, 64 :2}{30 :2} \rightarrow$$

$$\rightarrow \frac{\pi\, 64\sqrt{2}}{15} - \frac{\pi\, 32}{15} \rightarrow$$ Então temos :

$$\rightarrow \boxed{\frac{32\pi}{15} \cdot \left(2\sqrt{2} - 1\right)}$$